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Stabilizing the Hierarchical Basis by Approximate Wavelets, I: Theory

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This paper proposes a stabilization of the classical hierarchical basis (HB) method by modifying the HB functions using some computationally feasible approximate L^2-projections onto finite element spaces of relatively coarse levels. The corresponding multilevel additive and multiplicative algorithms give spectrally equivalent preconditioners, and one action of such a preconditioner is of optimal order computationally. The results are regularity-free for the continuous problem (second order elliptic) and can be applied to problems with rough coefficients and local refinement. © 1997 by John Wiley & Sons, Ltd. Numer. Linear Algebra Appl., Vol. 4, 103–126 (1997)

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1. Introduction

In this paper we are concerned with stabilizing the classical hierarchical basis (HB) introduced by Yserentant [29] (see also Bank, Dupont and Yserentant [4]) in the finite element application to second-order elliptic boundary value problems. The proposed method modifies the hierarchical basis functions by using some approximate L^2-projections on each

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level, yielding a basis which is a close relative to the well-known Battle–Lemarié wavelets [13].

For simplicity, we illustrate the idea for the elliptic problem which seeks $u \in H_0^1(\Omega)$ such that

$$-\nabla \cdot (a(x)\nabla u) = f(x) \qquad \text{in } \Omega \tag{1.1}$$

where Ω is an open bounded polygonal or polyhedral domain. The coefficient matrix $a = a(x)$ has bounded and measurable entries, and is assumed to be symmetric and positive definite over Ω. The equation (1.1) is discretized by the Galerkin method with continuous piecewise linear functions. We assume that the finite element triangulation $\mathcal{T}_h$ is constructed from a series of successive refinements described as follows. First let $\mathcal{T}_0$ be an initial coarse triangulation of Ω. Then, each $\mathcal{T}_k$, $k = 1, \cdots, J$, is obtained from $\mathcal{T}_{k-1}$ by breaking up each element of $\mathcal{T}_{k-1}$ into a couple of smaller, but congruent elements. Without loss of generality, this article will deal with the standard uniform refinement. The finite element space V_h employed in the Galerkin method corresponds to the partition of Ω on the finest level J so that $\mathcal{T}_h \equiv \mathcal{T}_J$. Denote by h_k the mesh size for the partition $\mathcal{T}_k$. Notice that $h_k = 2^{-k}h_0$. Let V_k denote the finite element space of continuous piecewise linear functions over $\mathcal{T}_k$. Finally, let $\mathcal{N}_k$ be the set of nodal points at level k which consists of all the vertices of elements of $\mathcal{T}_k$ and use the two-level hierarchical (direct) node-set decomposition, $\mathcal{N}_k = \mathcal{N}_k^{(1)} \cup \mathcal{N}_{k-1}$.

The stabilized basis functions are of the form $\psi_i^{(k)} \equiv (I - Q_{k-1}^a)\phi_i^{(k)}$ where $\phi_i^{(k)}$ are hierarchical basis functions at level k (i.e., associated with the node-set $\mathcal{N}_k^{(1)}$) and Q_{k-1}^a is an approximate L^2-projection onto the finite element space V_{k-1} with the understanding that $Q_{-1}^a = 0$. Our main result can be stated as follows:

Theorem 1.1. *(a) The set of functions $\{\psi_i^{(k)}\}_{i,k}$ forms a basis for the finite element space V_h. (b) For any $v \in V_h$, let*

$$v = \sum_{x_i \in \mathcal{N}_0} c_{0,i}\psi_i^{(0)} + \sum_{k=1}^{J} \sum_{x_i \in \mathcal{N}_k^{(1)}} c_{k,i}\psi_i^{(k)} \tag{1.2}$$

be its representation with respect to the approximate wavelet basis $\{\psi_i^{(k)}\}_{i,k}$ and define

$$|\!|\!| v |\!|\!|^2 = h_0^{d-2} \sum_{x_i \in \mathcal{N}_0} c_{0,i}^2 + \sum_{k=1}^{J} h_k^{d-2} \sum_{x_i \in \mathcal{N}_k^{(1)}} c_{k,i}^2 \tag{1.3}$$

where $d = 2$ or 3 according to the relation $\Omega \subset \mathbb{R}^d$. Assume that the operator Q_k^a satisfies

$$\|(Q_k - Q_k^a)v\|_0 \leq \tau \|Q_k v\|_0 \quad \text{forall} v \in L^2(\Omega) \tag{1.4}$$

Then, there exists a constant C_R such that

$$c_1 |\!|\!| v |\!|\!|^2 \leq \|v\|_1^2 \leq c_2 |\!|\!| v |\!|\!|^2$$

whenever $\tau < C_R^{-1}$. Here c_1 and c_2 are two absolute constants and $\|\cdot\|_1$ indicates the standard H^1-norm.

The constant C_R in Theorem 1.1 is given by the following estimate

$$\|(I_k - I_{k-1})v\|_0 \le C_R \|v\|_0 \quad \forall v \in V_k$$

Here I_s represents the standard nodal interpolation operator onto V_s.

As to the implementation, we shall construct preconditioners for the finite element discretization matrix $A^{(J)}$ by using the block-matrix approach employed by Bank, Dupont and Yserentant [4], Vassilevski [22], Axelsson and Vassilevski [2]. Details of this approach can be found in the survey paper by Vassilevski [24].

The mathematical theory for the stabilized hierarchical basis is based on the norm equivalence due to Oswald [19] (see also Dahmen and Kunoth [11], and Bornemann and Yserentant [5]) and the strengthened Cauchy inequality originated by Yserentant [29]. The argument adopted in this paper is a refinement of the algebraic (i.e., block-matrix) procedure from Axelsson and Vassilevski [2] (see also [22,23,24]). A similar block-matrix approach was later used by Griebel and Oswald [16].

It is interesting to note that the analysis in the spectral estimate for the multiplicative preconditioner is different from the technique first proposed in Bramble *et al*. [10] (see also Wang [27] and Vassilevski and Wang [25]). But the basic elements (see (a.i) and (a.ii) in section 4.2) for both approaches are the same. The new insight here is the perturbation analysis presented in section 5; its essence can be found in Lemma 5.1 and the estimate (5.9).

The results in the present paper can be applied to problems that require only the H^1-equivalent basis. The Stokes and elasticity equations in fluid dynamics and material science are two examples with this feature.

We now comment briefly on related approaches. A method, called pre-wavelet space decomposition, was reported by Kotyczka and Oswald [18] for two-dimensional regular meshes. For tensor product meshes pre-wavelet space decompositions were also investigated by Griebel and Oswald in [15]. The results in [18] and [15] have some restrictions on either the mesh or the analysis. In Stevenson [21] (which is an extension of [20]), essential progress was made toward more general meshes. More precisely, Stevenson proposed a direct wavelet-like multilevel decomposition on general meshes which exploits the discrete L^2-orthogonal decomposition $V_k = V_{k-1} \oplus V_k^1$, where V_k^1 admits basis functions that are linear combinations of three (standard nodal) basis functions of V_k. For recent results exploiting wavelets in the Galerkin method for solving partial differential equations, see Dahmen, Kunoth, and Urban [12].

The approach in the present paper is general and applicable to problems wherever the hierarchical decomposition of the finite element space exists with hierarchical components having a nodal basis, including spaces corresponding to non-uniformly refined meshes. The precise statement regarding the mesh non-uniformity can be found in Bornemann and Yserentant [5].

The paper is organized as follows. In section 2, we present an abstract framework of the algebraic multilevel preconditioning procedure which extends the two-level block matrix factorization method of Bank and Dupont [3] (see also Braess [6]). In section 3 we modify the hierarchical basis by using the exact L^2-projection operators. In section 4 we analyze the spectrum of the corresponding multiplicative preconditioner in the finite element application for second-order elliptic equations. In section 5 we present a computationally feasible modification of the hierarchical basis by using some approximate L^2-projections. A spectral estimate for the approximate wavelet preconditioners is established in section 5 as well. In section 6 we show that the stiffness matrix arising from the approximate wavelet basis is well-conditioned.

2. An abstract framework

In this section we describe a multilevel preconditioning technique for matrices in block structure. The technique was originated by Bank and Dupont in [3] and Braess [6] as a two-level procedure. Its analysis and multilevel extensions were later exploited by several researchers including Axelsson and Gustafsson [1], Bank, Dupont and Yserentant [4], and Vassilevski [22], [24].

Let $\mathbf{V}$ be a Euclidean space of dimension n equipped with the inner product $(\cdot,\cdot)$. Consider the problem of seeking $\mathbf{u} \in \mathbf{V}$ satisfying

$$A\mathbf{u} = \mathbf{b} \tag{2.1}$$

where $A = \{a_{ij}\}_{i,j=1}^{n}$ is a symmetric and positive definite matrix. The right-hand side vector $\mathbf{b}$ is given in $\mathbf{V}$.

Of interest in this paper, we assume that the condition number of A is large. Our objective is to find a good preconditioner for A. Then, some iterative methods (e.g., the Jacobi and conjugate gradient methods) can be employed to yield good approximations of (2.1) efficiently. This goal will be accomplished by transforming A to a matrix corresponding to an appropriately chosen basis of $\mathbf{V}$. The rest of this section is devoted to a detailed discussion of this preconditioning procedure.

Let $\mathcal{Y} = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ be a new basis for $\mathbf{V}$. Denote, for any $\mathbf{v} \in \mathbf{V}$, by $\widehat{\mathbf{v}} = (\widehat{v}_1, \widehat{v}_2, \ldots, \widehat{v}_n)^{\mathrm{T}}$ the co-ordinates of $\mathbf{v}$ with respect to the new basis $\mathcal{Y}$. Notice that $\mathbf{v} = \sum_{i=1}^{n} \widehat{v}_i \mathbf{y}_i$. The matrix $Y = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$ transforms the vector $\widehat{\mathbf{v}}$ to $\mathbf{v}$ as follows:

$$\mathbf{v} = Y\widehat{\mathbf{v}} \tag{2.2}$$

With the help from the transformation matrix Y, the problem (2.1) is equivalent to the seeking of $\widehat{\mathbf{u}}$ such that

$$\widehat{A}\widehat{\mathbf{u}} = \widehat{\mathbf{b}}, \qquad \text{with } \widehat{A} = Y^{\mathrm{T}}AY, \quad \widehat{\mathbf{b}} = Y^{\mathrm{T}}\mathbf{b} \tag{2.3}$$

If the transformed matrix $\widehat{A}$ is well conditioned, then a preconditioner B for A can be constructed by solving (2.3) approximately. More precisely, for any $\mathbf{d} \in \mathbf{V}$, the action $B^{-1}\mathbf{d}$ can be computed by the following procedure:

- first find $\widehat{\mathbf{d}} = Y^{\mathrm{T}}\mathbf{d}$,
- then solve $\widehat{A}\widehat{\mathbf{x}} = \widehat{\mathbf{d}}$ by some simple iterative method (e.g., the Jacobi method),
- denote by $\widehat{\mathbf{x}}$ the approximation from the preceding step and set $B^{-1}\mathbf{d} = \mathbf{Y}\widehat{\mathbf{x}}$.

The construction of such a desirable basis $\mathcal{Y}$ is often difficult in practical computations. In what follows of this section, we present an abstract framework which constructs $\mathcal{Y}$ using recursively the two-level technique of Bank and Dupont [3].

Assume that the linear space $\mathbf{V}$ can be decomposed as follows:

$$\mathbf{V}_J \equiv \mathbf{V} = \mathbf{V}_J^1 \oplus \mathbf{V}_J^2 \tag{2.4}$$

Here '$\oplus$' denotes the direct sum of subspaces. For simplicity of notation, we have also introduced a subscript 'J' since we intend to use successively the same procedure $J \geq 1$ times. Each subspace is assigned an appropriately-chosen basis,

$$\begin{aligned} \mathbf{V}_J^1 : &\quad \mathcal{Y}_1 = \{\mathbf{y}_j^1 \in \mathbf{V},\ j = 1, 2, \ldots, k_1\} \\ \mathbf{V}_J^2 : &\quad \mathcal{Y}_2 = \{\mathbf{y}_j^2 \in \mathbf{V},\ j = 1, 2, \ldots, k_2\} \end{aligned}$$

The sets of vectors from $\mathcal{Y}_1$ and $\mathcal{Y}_2$ forms a basis of $\mathbf{V}$. For any $\mathbf{v} \in \mathbf{V}$, let $\widehat{\mathbf{v}} = \begin{bmatrix} \widehat{\mathbf{v}}_1 \\ \widehat{\mathbf{v}}_2 \end{bmatrix}$ be the co-ordinates with respect to the new basis, with $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$ being the components in $\mathbf{V}_J^1$ and $\mathbf{V}_J^2$, respectively. The transformation matrix Y is, therefore, decomposed as $Y = [Y_1, Y_2]$, satisfying

$$Y_1\widehat{\mathbf{v}}_1 + Y_2\widehat{\mathbf{v}}_2 = \mathbf{v} \tag{2.5}$$

With the above partition, one obtains the following block-form for $\widehat{A}$:

$$\widehat{A} = [Y_1, Y_2]^{\mathrm{T}}\, A\, [Y_1, Y_2] = \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_{22} \end{bmatrix} \tag{2.6}$$

where

$$\begin{aligned} \widehat{A}_{11} &= Y_1^{\mathrm{T}} A Y_1, \quad & \widehat{A}_{12} &= Y_1^{\mathrm{T}} A Y_2 \\ \widehat{A}_{21} &= Y_2^{\mathrm{T}} A Y_1, \quad & \widehat{A}_{22} &= Y_2^{\mathrm{T}} A Y_2 \end{aligned}$$

It would be unrealistic to assume that the matrix $\widehat{A}$ is well conditioned. However, the submatrix $\widehat{A}_{11}$ might become well conditioned for an appropriately chosen decomposition (2.4). Thus, we make the following assumption:

A1. *There exists a direct decomposition (2.4) and a basis for $\mathbf{V}_J^1$ so that the submatrix $\widehat{A}_{11}$ is well-conditioned.*

The submatrix $A_{J-1} \equiv \widehat{A}_{22}$ is the block of A on $\mathbf{V}_J^2$, which may not be well conditioned. This difficulty can be overcome by repeating the above procedure, now applied to $\mathbf{V}_{J-1} \equiv \mathbf{V}_J^2$ and the block A_{J-1}. Therefore, the procedure will create a direct decomposition:

$$\mathbf{V} = \mathbf{V}_J^1 \oplus \mathbf{V}_{J-1}^1 \oplus \cdots \oplus \mathbf{V}_1^1 \oplus \mathbf{V}_0 \tag{2.7}$$

so that the restriction of A to each subspace $\mathbf{V}_j^1$ gives well–conditioned matrices. With $\mathbf{V}_j = \mathbf{V}_j^1 \oplus \mathbf{V}_j^2$ and $\mathbf{V}_{j-1} \equiv \mathbf{V}_j^2$, where $\mathbf{V}_J = \mathbf{V}$ and $\mathbf{V}_0$ is a space of relatively small dimension, the direct decomposition (2.7) can be written recursively as follows:

$$\mathbf{V}_j = \mathbf{V}_j^1 \oplus \mathbf{V}_{j-1}, \quad j = J, J-1, \ldots, 1 \tag{2.8}$$

We emphasize that each of $\mathbf{V}_j^1$ and $\mathbf{V}_{j-1}$ is equipped with an appropriately-chosen basis which together form a new basis for $\mathbf{V}_j$.

We now construct a sequence of matrices $\{A_j\}$; each can be considered as a linear operator on the subspace $\mathbf{V}_j$ for $j = 1, 2, \ldots, J$. Assume that A_j has been constructed on $\mathbf{V}_j$. Let $\widehat{A}_j$ be the representation of A_j with respect to the new basis provided by (2.8) and the given bases of $\mathbf{V}_j^1$ and $\mathbf{V}_{j-1}$ $(= \mathbf{V}_j^2)$. Similar to (2.6), the matrix $\widehat{A}_j$ has the following block structure:

$$\widehat{A}_j = \begin{bmatrix} \widehat{A}_{11}^{(j)} & \widehat{A}_{12}^{(j)} \\ \widehat{A}_{21}^{(j)} & \widehat{A}_{22}^{(j)} \end{bmatrix} \tag{2.9}$$

from which one defines $A_{j-1} \equiv \widehat{A}_{22}^{(j)}$.

The matrix $\widehat{A}_j$ admits the following standard block-Cholesky factorization,

$$\widehat{A}_j = \begin{bmatrix} \widehat{A}_{11}^{(j)} & 0 \\ \widehat{A}_{21}^{(j)} & A_{j-1} - \widehat{A}_{21}^{(j)} \left(\widehat{A}_{11}^{(j)}\right)^{-1} \widehat{A}_{12}^{(j)} \end{bmatrix} \begin{bmatrix} I & \left(\widehat{A}_{11}^{(j)}\right)^{-1} \widehat{A}_{12}^{(j)} \\ 0 & I \end{bmatrix} \tag{2.10}$$

For $j = 1, 2, \ldots, J$, let $\widehat{B}_{11}^{(j)}$ be preconditioners to $\widehat{A}_{11}^{(j)}$ satisfying some properties to be specified later (e.g., the relation (3.13) in section 3). By dropping the term $\widehat{A}_{21}^{(j)} \left(\widehat{A}_{11}^{(j)}\right)^{-1} \widehat{A}_{12}^{(j)}$ in (2.10), we have a preconditioner B_p for A from the following routine inductive procedure (see Vassilevski [22,23,24] for more information).

Algorithm 2.1. *Multiplicative preconditioner $B_p \equiv B_{J,p}$ First set $B_{0,p} = A_0$. Assume that a preconditioner $B_{j-1,p}$ for A_{j-1} has been constructed. Obtain one for A_j as follows:*

- *Set*

$$\widehat{B}_{j,p} \equiv \begin{bmatrix} \widehat{B}_{11}^{(j)} & 0 \\ \widehat{A}_{21}^{(j)} & B_{j-1,p} \end{bmatrix} \begin{bmatrix} I & \widehat{B}_{11}^{(j)^{-1}} \widehat{A}_{12}^{(j)} \\ 0 & I \end{bmatrix} \tag{2.11}$$

- *Get a preconditioner $B_{j,p}$ from $\widehat{B}_{j,p}$ by changing bases. More precisely, the preconditioner for A_j is determined by the equation $\widehat{B}_{j,p} = Y^{\mathrm{T}} B_{j,p} Y$, or $B_{j,p}^{-1} = Y \widehat{B}_{j,p}^{-1} Y^{\mathrm{T}}$ (see (2.6) for details).*

The multiplicative preconditioner B_p was constructed from the symmetric block Gauss–Seidel approximation of (2.10) using preconditioners of $\widehat{A}_{11}^{(j)}$ and A_{j-1}. If $\widehat{A}_j$ is approximated by its block-diagonal part in (2.9), then an additive preconditioner for A is possible.

Algorithm 2.2. *Additive preconditioner $B_a \equiv B_{J,a}$ First set $B_{0,a} = A_0$. Assume the existence of a preconditioner $B_{j-1,a}$ for A_{j-1}. Construct one for A_j as follows:*

- *Set*

$$\widehat{B}_{j,a} \equiv \begin{bmatrix} \widehat{B}_{11}^{(j)} & 0 \\ 0 & B_{j-1,a} \end{bmatrix} \tag{2.12}$$

- *Obtain a preconditioner $B_{j,a}$ from $\widehat{B}_{j,a}$ by changing bases. More precisely, the preconditioner of A_j is determined by the equation $\widehat{B}_{j,a} = Y^{\mathrm{T}} B_{j,a} Y$ or $B_{j,a}^{-1} = Y \widehat{B}_{j,a}^{-1} Y^{\mathrm{T}}$.*

Implementations of the additive and multiplicative preconditioners rely on the transformation matrices Y_j among the subspaces in the decomposition. Note that we have defined $\widehat{B}_{j,p}$ based on $B_{j-1,p}$ and, in the implementation, we will need the inverse actions of $B_{j,p}$. Based on the identity $B_{j,p}^{-1} = Y \widehat{B}_{j,p}^{-1} Y^{\mathrm{T}}$ we see that these actions are available assuming by induction that the actions of $B_{j-1,p}^{-1}$ are computable. Note also that the inverse actions of

Y and Y^{T} are not needed. The same argument applies for the additive preconditioner $B_{j,a}$. Details can be found from the second part of this work [26].

It should be pointed out that the decomposition (2.7) must be known prior to the implementation. Such a decomposition can be constructed by using various techniques which are often problem–dependent. In particular, one might be able to obtain a computationally feasible decomposition (2.7) by using properties of the matrices A_j only, yielding methods of algebraic multigrid-type. However, the following four sections shall be devoted to an investigation of (2.7) for the finite element discretization of (1.1) on *structured grids* which are obtained by a series of successive (possibly local) refinements for a given initial coarse triangulation of the physical domain as sketched in section 1.

3. Wavelet-modified HB preconditioners

We now return to the model problem (1.1) which is discretized by the Galerkin method as described in section 1. Let $\mathcal{N}_k$ be the set of nodal points at level k which consists of all the vertices of elements of $\mathcal{T}_k$. Recall that the refinement procedure generates a sequence of nested spaces: $V_0 \subset V_1 \subset \cdots \subset V_J$.

3.1. The hierarchical basis

Each finite element space V_k has a set of nodal (Lagrangian) basis:

$$V_k = \text{span}\,\{\phi_i^{(k)} \;:\; x_i \in \mathcal{N}_k\}$$

defined by $\phi_i^{(k)}(x_j) = \delta_{ij}$ where δ_{ij} is the standard Kronecker symbol. Let $\mathcal{N}_k^{(1)} = \mathcal{N}_k \backslash \mathcal{N}_{k-1}$ be the set of new nodal points at level k, and

$$V_k^{(1)} = \text{span}\{\phi_i^{(k)} \;:\; x_i \in \mathcal{N}_k^{(1)}\} \tag{3.1}$$

One then has the following direct decomposition:

$$V_k = V_k^{(1)} \oplus V_{k-1} \tag{3.2}$$

which is an analogue of (2.8). The subspace V_{k-1} is equipped with the standard nodal basis over the finite element partition $\mathcal{T}_{k-1}$.

A successive use of (3.2) yields the following HB decomposition for $V \equiv V_J$:

$$V = V_J^{(1)} \oplus V_{J-1}^{(1)} \oplus \ldots \oplus V_1^{(1)} \oplus V_0 \tag{3.3}$$

The classical multilevel hierarchical basis method (see Yserentant [29] and Bank, Dupont and Yserentant [4]) is a preconditioning technique based on (3.3). In particular, the method outlined in section 2 can be applied to yield some multiplicative and additive preconditioners for the global stiffness matrix A.

The difficulty with the hierarchical basis method is that the corresponding preconditioners are not spectrally equivalent to the original matrix. This is technically due to the fact that the interpolation operator $I_k : V \to V_k$, defined by $I_k v = \sum_{x_i \in \mathcal{N}_k} v(x_i)\phi_i^{(k)}$, is not bounded in the H^1-norm uniformly with respect to the difference $J - k \to \infty$ or the ratio of the mesh sizes h_k/h_J.

3.2. *Modified hierarchical bases*

Here we propose a general modification for the hierarchical basis. Specific examples with improved preconditioners will be discussed in next two sections.

Let M_j be bounded linear operators from $L^2(\Omega)$ to the finite element spaces V_j for $j = 0, \cdots, J$. The boundedness is considered as an operator from $L^2(\Omega) \to L^2(\Omega)$. Consider the following modification of $V_k^{(1)}$:

$$V_k^1 = (I - M_{k-1})V_k^{(1)}, \qquad k = 1, 2, \cdots, J$$

Lemma 3.1. *For each hierarchical basis function $\phi_i^{(k)} \in V_k^{(1)}$, let $\psi_i^{(k)} = (I - M_{k-1})\phi_i^{(k)}$. Then,*

$$\Gamma_k = \{\psi_i^{(k)} : \ \forall x_i \in \mathcal{N}_k^{(1)}\} \tag{3.4}$$

forms a basis of V_k^1. Moreover, the following decompositions hold:

$$\begin{aligned} V_k &= V_k^1 \oplus V_{k-1} \\ V &= V_J^1 \oplus V_{J-1}^1 \oplus \cdots \oplus V_1^1 \oplus V_0 \end{aligned} \tag{3.5}$$

Proof First we show that Γ_k is a set of linearly independent functions. Let $\{\alpha_i\}$ be real numbers such that

$$\sum_{x_i \in \mathcal{N}_k^{(1)}} \alpha_i \psi_i^{(k)}(x) = 0 \qquad \forall x$$

With $\phi = \sum_{x_i \in \mathcal{N}_k^{(1)}} \alpha_i \phi_i^{(k)} \in V_k^{(1)}$, the above leads to

$$\phi(x) - M_{k-1}\phi(x) = 0 \qquad \forall x$$

It follows that $\phi = M_{k-1}\phi \in V_{k-1} \cap V_k^{(1)} = \{0\}$. Thus, we obtain $\phi \equiv 0$ which implies $\alpha_i = 0$ for all i. This shows that Γ_k is a set of linearly independent functions.

Next since $\dim(V_k^1) \leq \dim(V_k^{(1)})$ and Γ_k is a linearly independent set, then $\dim(V_k^1) = \dim(V_k^{(1)})$ and Γ_k forms a basis for V_k^1. Similar arguments can be applied to show that $V_k^1 \cap V_{k-1} = \{0\}$, which verifies the validity of the first equality in (3.5). The second one in (3.5) is merely a by-product of the first. ■

3.3. *Wavelet-modified hierarchical bases*

Let $Q_k : L^2(\Omega) \to V_k$ be the L^2-projection defined by

$$(Q_k v, \phi) = (v, \phi) \qquad \forall \phi \in V_k \tag{3.6}$$

where $(.,.)$ stands for the standard $L^2(\Omega)$-inner product.

With $M_k = Q_k$, one obtains from Lemma 3.1 a modification of the hierarchical basis. It follows from the previous section that the modified V_k^1 is given by

$$V_k^1 \equiv (I - Q_{k-1})V_k^{(1)} = (I - Q_{k-1})V_k = (Q_k - Q_{k-1})V \tag{3.7}$$

Therefore, the subspaces V_k^1 are mutually orthogonal to each other in $L^2(\Omega)$. This modified hierarchical basis shall be called *wavelet basis* (see [13] for more information). The preconditioning methods discussed in section 2 are then applicable to the wavelet decomposition, yielding some *wavelet preconditioners* for the stiffness matrix A.

We now describe an equivalent, but implementation-oriented, approach for the construction of the wavelet preconditioners. To this end, let $a(u, v) \equiv \int_\Omega a(x)\nabla u \cdot \nabla v \mathrm{d}x$ be the bilinear form associated with the elliptic operator in the problem (1.1). The following discretizations of the elliptic operator are needed:

- The discretization (solution) operator $A^{(k)} : V_k \to V_k$ at level k defined by

$$(A^{(k)}v, \psi) = a(v, \psi) \quad \forall v, \psi \in V_k \tag{3.8}$$

  Denote by λ_k the largest eigenvalue of $A^{(k)}$.
- The discretization operator $A_{11}^{(k)} : V_k^1 \to V_k^1$ in the subspace V_k^1:

$$(A_{11}^{(k)}v, \psi) = a(v, \psi) \quad \forall v, \psi \in V_k^1 \tag{3.9}$$

  Let $\lambda_{k;\,\max}^1$ and $\lambda_{k;\,\min}^1$ be the largest and smallest eigenvalues of $A_{11}^{(k)}$.
- The communication operators $A_{12}^{(k)} : V_{k-1} \to V_k^1$ and $A_{21}^{(k)} : V_k^1 \to V_{k-1}$,

$$\begin{aligned}(A_{12}^{(k)}\tilde{\psi}, v^1) &= a(v^1, \tilde{\psi}) \qquad \forall v^1 \in V_k^1, \tilde{\psi} \in V_{k-1}\\ (A_{21}^{(k)}v^1, \tilde{\psi}) &= a(v^1, \tilde{\psi}) \qquad \forall v^1 \in V_k^1,\ \tilde{\psi} \in V_{k-1}\end{aligned} \tag{3.10}$$

  Note that $A_{12}^{(k)}$ is the L^2–adjoint of $A_{21}^{(k)}$.

Since the decomposition $V_k = V_k^1 \oplus V_{k-1}$ is direct, one has the following two-by-two block form for the operator $A^{(k)}$:

$$A^{(k)} = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A^{(k-1)} \end{bmatrix} \begin{matrix} \} & V_k^1 \\ \} & V_{k-1} \end{matrix} \tag{3.11}$$

The decomposition (3.11) can be seen as follows. For any $v, \psi \in V_k$ decomposed as $v = v^1 + \tilde{v}$ and $\psi = \psi^1 + \tilde{\psi}$, where $v^1 = (I - Q_{k-1})v \in V_k^1$, $\psi^1 = (I - Q_{k-1})\psi \in V_k^1$ and $\tilde{v} = Q_{k-1}v \in V_{k-1}$, $\tilde{\psi} = Q_{k-1}\psi \in V_{k-1}$, one has

$$\begin{aligned}&(A^{(k)}v, \psi) = a(v, \psi) = a(v^1 + \tilde{v}, \psi^1 + \tilde{\psi})\\ &= a(v^1, \psi^1) + a(\tilde{v}, \psi^1) + a(v^1, \tilde{\psi}) + a(\tilde{v}, \tilde{\psi})\\ &= (A_{11}^{(k)}v^1, \psi^1) + (A_{12}^{(k)}\tilde{v}, \psi^1) + (A_{21}^{(k)}v^1, \tilde{\psi}) + (A^{(k-1)}\tilde{v}, \tilde{\psi})\\ &= \left(\begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A^{(k-1)} \end{bmatrix}\begin{bmatrix} v^1 \\ \tilde{v} \end{bmatrix}, \begin{bmatrix} \psi^1 \\ \tilde{\psi} \end{bmatrix}\right)\end{aligned} \tag{3.12}$$

To construct the wavelet-modified HB preconditioners, we assume the existence of some given approximations $B_{11}^{(k)}$, which are symmetric and positive definite in V_k^1, to the operators $A_{11}^{(k)}$, $k = 1, 2, \ldots, J$. We also assume the validity of the following spectral equivalence:

$$(A_{11}^{(k)}v^1, v^1) \le (B_{11}^{(k)}v^1, v^1) \le (1 + b_1)(A_{11}^{(k)}v^1, v^1) \quad \forall v^1 \in V_k^1 \tag{3.13}$$

Here $b_1 > 0$ is an absolute constant independent of k and J. In practical implementation, $B_{11}^{(k)}$ can simply be chosen as a diagonal matrix.

Algorithm 3.1. *Wavelet-modified multiplicative HB preconditioner Let $B^{(0)} = A^{(0)}$. For $k = 1, \dots, J$ define*

$$B^{(k)} = \begin{bmatrix} B_{11}^{(k)} & 0 \\ A_{21}^{(k)} & B^{(k-1)} \end{bmatrix} \begin{bmatrix} I & B_{11}^{(k)^{-1}} A_{12}^{(k)} \\ 0 & I \end{bmatrix}$$

Note that one solution with $B^{(k)}$ requires two solutions with each of the approximations $B_{11}^{(s)}$, $s = 1, 2, \dots, k$, one action with each of $A_{12}^{(s)}$ and $A_{21}^{(s)}$, $s = 1, 2, \dots, k$ and a coarse-grid solution with $A^{(0)}$. The additive preconditioner can be defined as in (2.12).

We point out that the wavelet basis functions $\psi_i^{(k)} \equiv (I - Q_{k-1})\phi_i^{(k)}$ are not locally supported in Ω which causes a difficulty in implementing the wavelet basis. To overcome this difficulty, Jaffard [17] suggested the use of FFT in order to compute the expansion of functions. But this approach imposes some restriction on the mesh structure. There has been a different approach by Oswald (see section 4.2 in [19]) which, for some special cases of the partition, constructs a new basis for V_k^1 with local support. In conclusion, the subspace V_k^1 is not a desirable choice in the multilevel preconditioning method based on the matrix blocks $A_{11}^{(k)}$, $A_{12}^{(k)}$, and $A_{21}^{(k)}$. A remedial procedure is presented in section 5 in which the subspace V_k^1 will be replaced by a small perturbation of itself in order to have a set of locally-supported basis.

4. Spectral analysis

Here we analyze the preconditioner $B^{(k)}$ by refining the argument of Vassilevski [23] and [22].

4.1. *A general result*

Let $E^{(k)} \equiv B^{(k)} - A^{(k)}$ be the difference between $A^{(k)}$ and its preconditioner $B^{(k)}$. For any $v \in V_k$ with $v = v^1 + \tilde{v}$, where $v^1 \in V_k^1$ and $\tilde{v} \in V_{k-1}$, one has from (3.12) and some elementary computation that

$$\begin{aligned} (E^{(k)}v, v) &= (B^{(k)}v, v) - (A^{(k)}v, v) \\ &= ((B_{11}^{(k)} - A_{11}^{(k)})v^1, v^1) + (E^{(k-1)}\tilde{v}, \tilde{v}) + (B_{11}^{(k)^{-1}} A_{12}^{(k)}\tilde{v}, A_{12}^{(k)}\tilde{v}) \end{aligned} \tag{4.1}$$

The operator $E^{(k)}$ is positive semi-definite. In fact, this is true for $k = 0$ because $E^{(0)} = 0$. Assume that $E^{(s)}$ is positive semi–definite on $s < k$. It follows from (4.1) and (3.13) that $(E^{(k)}v, v) \geq 0$ for all $v \in V_k$.

An upper bound for $E^{(k)}$ can be derived by using (3.13) and two inequalities to be specified later. First, using (3.13) in (4.1) one obtains

$$(E^{(k)}v, v) \leq b_1(A_{11}^{(k)}v^1, v^1) + (E^{(k-1)}\tilde{v}, \tilde{v}) + (B_{11}^{(k)^{-1}} A_{12}^{(k)}\tilde{v}, A_{12}^{(k)}\tilde{v})$$

In general, if $v^{(s)} \in V_s$ has the decomposition

$$v^{(s)} = v^{(s)^1} + v^{(s-1)}, \quad v^{(s)^1} \in V_s^1,\ v^{(s-1)} \in V_{s-1} \tag{4.2}$$

then

$$\begin{aligned}&(E^{(s)}v^{(s)}, v^{(s)}) - (E^{(s-1)}v^{(s-1)}, v^{(s-1)})\\ &\le b_1\,(A_{11}^{(s)}v^{(s)^1}, v^{(s)^1}) + (B_{11}^{(s)^{-1}}A_{12}^{(s)}v^{(s-1)}, A_{12}^{(s)}v^{(s-1)})\end{aligned}$$

Summing over s yields (with $v = v^{(k)}$)

$$(E^{(k)}v, v) \le b_1 \sum_{s=1}^{k}(A_{11}^{(s)}v^{(s)^1}, v^{(s)^1}) + \sum_{s=1}^{k}(B_{11}^{(s)^{-1}}A_{12}^{(s)}v^{(s-1)}, A_{12}^{(s)}v^{(s-1)})$$

Thus, an upper bound can be derived for $E^{(k)}$ if the following two inequalities can be established. *There exist two constants ϱ_1 and ϱ_2 both independent of k such that*

$$\sum_{s=1}^{k}(A_{11}^{(s)}v^{(s)^1}, v^{(s)^1}) \le \varrho_1(A^{(k)}v, v) \tag{4.3}$$

and

$$\sum_{s=1}^{k}(B_{11}^{(s)^{-1}}A_{12}^{(s)}v^{(s-1)}, A_{12}^{(s)}v^{(s-1)}) \le \varrho_2(A^{(k)}v, v) \tag{4.4}$$

for all $v \in V_k$.

We emphasize that $v^{(s)^1}$ and $v^{(s-1)}$ are determined by (4.2) with $v^{(k)} = v$. To summarize, the following result has been proved:

Theorem 4.1. *If (3.13), (4.3) and (4.4) hold true, then the following is valid for the preconditioner $B^{(k)}$*

$$(A^{(k)}v, v) \le (B^{(k)}v, v) \le (b_1\varrho_1 + \varrho_2)(A^{(k)}v, v) \qquad \forall v \in V_k \tag{4.5}$$

The spectral estimate (4.5) is a general result for the multiplicative preconditioner introduced in section 2. The result is based on three inequalities which must be established for each spatial decomposition (2.7).

4.2. An application to the wavelet basis

Our objective is to establish the inequalities (4.3) and (4.4). The argument is based on two fundamental inequalities in the multilevel theory. Namely, there exists a constant σ independent of k satisfying

(a.i) $\|Q_0 v\|_1^2 + \sum_{j=1}^{k} h_j^{-2}\|(Q_j - Q_{j-1})v\|_0^2 \le \sigma\|v\|_1^2 \qquad \forall v \in V_k$

and

(a.ii) $\left|a(\psi_i, \psi_j)\right|^2 \le \sigma\delta^{2(j-i)}h_j^{-2}a(\psi_i, \psi_i)\|\psi_j\|_0^2, \quad \forall \psi_i \in V_i,\ \psi_j \in V_j,\ j \ge i$, where $\delta \in (0, 1)$ is a constant given by the upper bound of the ratio h_i/h_{i-1} for $i = 1, \ldots, J$.

Here and in what follows, $\|\cdot\|_s$ denotes the norm in the Sobolev space $H^s(\Omega)$ for $s = 0, 1$.

The inequality (a.i) was proved in Oswald [19] (see also [5,11]) and (a.ii) was originally seen in Yserentant [30] (see also [7,25,28]). The following result confirms (4.3).

Lemma 4.1. *Assume that (a.i) holds true. There exists a constant C such that*

$$\sum_{s=1}^{k}(A_{11}^{(s)}v^{(s)^1}, v^{(s)^1}) \leq C\ (A^{(k)}v, v) \qquad \forall v \in V_k \tag{4.6}$$

Proof For the wavelet basis decomposition, one has from (4.2) that

$$v^{(s-1)} = Q_{s-1}v^{(s)} \quad \text{and} \quad v^{(s)^1} = v^{(s)} - v^{(s-1)}$$

Thus,

$$v^{(s)} = Q_s Q_{s+1} \cdots Q_k v = Q_s v,\ \ v^{(s)^1} = (Q_s - Q_{s-1})v$$

It follows from the inverse inequality and (a.i) that

$$\begin{aligned}\sum_{s=1}^{k}\ (A_{11}^{(s)}v^{(s)^1}, v^{(s)^1}) &\leq C\sum_{s=1}^{k} h_s^{-2}\|v^{(s)^1}\|_0^2 \\ &= C\sum_{s=1}^{k} h_s^{-2}\|(Q_s - Q_{s-1})v\|_0^2 \leq Ca(v, v)\end{aligned}$$

which completes the proof of the lemma. ■

The following result will be used to verify the validity of (4.4).

Lemma 4.2. *If (a.i) and (a.ii) hold true, then there exists a constant C such that*

$$\sum_{s=1}^{k} h_s^2\|A_{12}^{(s)}v^{(s-1)}\|_0^2 \leq Ca(v, v) \qquad \forall v \in V_k \tag{4.7}$$

Proof First by using (3.10) and (3.8) one has

$$\|A_{12}^{(s)}v^{(s-1)}\|_0^2 = a(v^{(s-1)}, A_{12}^{(s)}v^{(s-1)}) = (A^{(s)}v^{(s-1)}, A_{12}^{(s)}v^{(s-1)})$$

Thus, by using the Schwarz inequality

$$\|A_{12}^{(s)}v^{(s-1)}\|_0^2 \leq \|A^{(s)}v^{(s-1)}\|_0^2$$

Introduce the operator $T_j = h_j^2 A^{(j)}$. Hence,

$$h_s^2\|A_{12}^{(s)}v^{(s-1)}\|_0^2 \leq h_s^2\|A^{(s)}v^{(s-1)}\|_0^2 = a(T_s v^{(s-1)}, v^{(s-1)}) \tag{4.8}$$

By using the decomposition

$$v^{(s-1)} \equiv Q_{s-1}v = \sum_{j=0}^{s-1}(Q_j - Q_{j-1})v \equiv \sum_{j=0}^{s-1} v^{(j)^1}$$

one obtains

$$a(T_s v^{(s-1)}, v^{(s-1)}) = \sum_{j=0}^{s-1} a(T_s v^{(s-1)}, v^{(j)^1}) \tag{4.9}$$

Now using the strengthened Cauchy inequality (a.ii) (note that $j < s$),

$$\begin{aligned} \left| a(T_s v^{(s-1)}, v^{(j)^1}) \right|^2 &\le \sigma^2 \delta^{2(s-j)} h_s^{-2} a(v^{(j)^1}, v^{(j)^1}) \| T_s v^{(s-1)} \|_0^2 \\ &= \sigma^2 \delta^{2(s-j)} h_s^2 a(v^{(j)^1}, v^{(j)^1}) \ \| A^{(s)} v^{(s-1)} \|_0^2 \\ &= \sigma^2 \delta^{2(s-j)} \ a(v^{(j)^1}, v^{(j)^1}) \ a(T_s v^{(s-1)}, v^{(s-1)}) \end{aligned}$$

Therefore, substituting the above into (4.9) yields

$$a(T_s v^{(s-1)}, v^{(s-1)}) \le \sigma^2 \left[\sum_{j=0}^{s-1} \delta^{s-j} \left[a(v^{(j)^1}, v^{(j)^1}) \right]^{\frac{1}{2}} \right]^2$$

Applying the Cauchy–Schwarz inequality one obtains

$$a(T_s v^{(s-1)}, v^{(s-1)}) \le \sigma^2 \frac{\delta}{1-\delta} \sum_{j=0}^{s-1} \delta^{s-j} a(v^{(j)^1}, v^{(j)^1})$$

Summing over s leads to the following

$$\begin{aligned} \sum_{s=1}^{k} a(T_s v^{(s-1)}, v^{(s-1)}) &\le \sigma^2 \frac{\delta}{1-\delta} \sum_{s=1}^{k} \sum_{j=0}^{s-1} \delta^{s-j} a(v^{(j)^1}, v^{(j)^1}) \\ &\le \sigma^2 \left(\frac{\delta}{1-\delta} \right)^2 \sum_{j=0}^{k-1} a(v^{(j)^1}, v^{(j)^1}) \end{aligned}$$

which together with (4.6) and (3.9) implies

$$\sum_{s=1}^{k} a(T_s v^{(s-1)}, v^{(s-1)}) \le C(A^{(k)} v, v) \qquad \forall v \in V_k$$

The lemma is thus verified by combining the above inequality with (4.8). ∎

The following useful result, which is a reformulation of **A1** in section 2, will be proved in section 6.1.

Lemma 4.3. *If $\lambda^1_{k,\,\min}$ and $\lambda^1_{k,\,\max}$ are the smallest and largest eigenvalues of $A^{(k)}_{11}$, then there exist constants C_1 and C_2 both independent of h_k such that,*

$$C_1 h_k^{-2} \le \lambda^1_{k;\,\min} \le \lambda^1_{k;\,\max} \le C_2 h_k^{-2}$$

Consequently, the matrix $A^{(k)}_{11}$ is well conditioned.

We are now in a position to verify the inequality (4.4). Observe that from Lemma 4.3 the matrix $A^{(s)}_{11}$ is well conditioned. Thus, one may choose a diagonal preconditioner $B^{(s)}_{11} =$

$\alpha h_s^{-2} I$ for the matrix $A_{11}^{(s)}$. Here α is a parameter which should be adjusted so that (3.13) is satisfied for some b_1. With the above selection of $B_{11}^{(s)}$, it is trivial to see that

$$\sum_{s=1}^{k} (B_{11}^{(s)^{-1}} A_{12}^{(s)} v^{(s-1)}, A_{12}^{(s)} v^{(s-1)}) \le C \sum_{s=1}^{k} h_s^2 \|A_{12}^{(s)} v^{(s-1)}\|_0^2 \le C(A^{(k)} v, v)$$

where we have used (4.7) in the last inequality. The general case for $B_{11}^{(s)}$ satisfying (3.13) can be treated similarly since $B_{11}^{(s)^{-1}}$ is spectrally equivalent to $\alpha^{-1} h_s^2 I$. This verifies the validity of (4.4). The result can be summarized as follows:

Theorem 4.2. *If the inequalities (a.i) and (a.ii) hold true, then there exists a constant $C > 0$ independent of k such that*

$$(A^{(k)} v, v) \le (B^{(k)} v, v) \le C(A^{(k)} v, v) \qquad \forall v \in V_k$$

for $k = 0, 1, \ldots, J$. The constant C depends only on b_1 from (3.13), δ from (a.ii), and σ from (a.i), (a.ii).

It should be pointed out that the validity of (a.ii) relies on some regularity assumption for the coefficient matrix $a = a(x)$ of (1.1). Thus, the estimate (4.7) is not known for the elliptic equation (1.1) with arbitrary $a(x)$. Without assuming (a.ii), one can derive the following straightforward sub-optimal estimate:

Lemma 4.4. *If (a.i) holds true, then there exists a constant C such that*

$$\sum_{s=1}^{k} h_s^2 \|A_{12}^{(s)} v^{(s-1)}\|_0^2 \le Ck\, a(v, v) \qquad \forall v \in V_k \tag{4.10}$$

Proof From (4.8) we get

$$\sum_{s=1}^{k} h_s^2 \|A_{12}^{(s)} v^{(s-1)}\|_0^2 \le \sum_{s=1}^{k} a(T_s v^{(s-1)}, v^{(s-1)}) \tag{4.11}$$

Since $T_s = h_s^2 A^{(s)}$ and the largest eigenvalue of $A^{(s)}$ is proportional to h_s^{-2}, then there exists a constant C such that

$$a(T_s v^{(s-1)}, v^{(s-1)}) \le C\|v^{(s-1)}\|_1^2 = C\|Q_{s-1} v\|_1^2 \le C\|v\|_1^2 \tag{4.12}$$

where we have used the fact that the L^2 projection operator Q_{s-1} is bounded in $H^1(\Omega)$. Substituting (4.12) into (4.11) yields (4.10). ∎

Theorem 4.3. *If the inequality (a.i) holds true, then there exists a constant $C > 0$ independent of k such that*

$$(A^{(k)} v, v) \le (B^{(k)} v, v) \le C(1 + k)\, (A^{(k)} v, v) \qquad \forall v \in V_k$$

for $k = 0, 1, \ldots, J$. The constant C depends only on b_1 from (3.13), σ from (a.i), and the H^1-norm of the L^2-projection operator Q_s for $s = 1, \cdots, k$.

Proof It suffices to verify the conditions of Theorem 4.1. The inequalities (3.13) and (4.3) were already proved in Lemmas 4.3 and 4.1. The validity of (4.4) was concluded by Lemma 4.4 with $\varrho_2 = Ck$. ■

Based on a more refined analysis due to Griebel and Oswald [16], the following result can be proved:

Theorem 4.4. *If the inequality (a.i) holds true, then there exists a constant $C > 0$ independent of k such that*

$$(A^{(k)}v, v) \le (B^{(k)}v, v) \le C(1 + \log_2(1 + k))\,(A^{(k)}v, v) \qquad \forall v \in V_k$$

for $k = 0, 1, \ldots, J$. The constant C depends only on b_1 from (3.13), σ from (a.i), and the H^1-norm of the L^2-projection operator Q_s for $s = 1, \ldots, k$.

5. Approximate wavelet bases and preconditioners

In this section we present a computationally feasible modification of the hierarchical basis by approximating the L^2-projections Q_s. To this end, let Q_s^a be a bounded linear operator that approximates the exact L^2-projection Q_s in the sense that there exists a small (but fixed) $\tau > 0$ satisfying

$$\|(Q_s^a - Q_s)v)\|_0 \le \tau \|Q_s v\|_0 \qquad \forall v \in V \tag{5.1}$$

In practical computation, the approximation operator Q_s^a is given as a polynomial of the Gram matrix associated with V_s. Details will be given in the second part of the paper.

With M_j replaced by Q_j^a in section 3.2, one obtains a modified hierarchical basis as a perturbation of the wavelet basis. Such a basis is called *approximate wavelet basis* in this paper. It follows from Lemma 3.1 that the approximate wavelet basis is given by

$$\Gamma_k = \{\psi_i^{(k)} \equiv (I - Q_{k-1}^a)\phi_i^{(k)} : \quad \forall x_i \in \mathcal{N}_k^{(1)},\ k = 0, 1, \cdots, J\}$$

where $\{\phi_i^{(k)}\}$ is the set of the hierarchical basis functions. Also, one has from (3.5) that

$$V = V_J^1 \oplus V_{J-1}^1 \oplus \cdots \oplus V_1^1 \oplus V_0$$

with $V_k^1 = (I - Q_{k-1}^a)V_k^{(1)}$. Each subspace V_k^1 is equipped with the following basis:

$$\Gamma_k = \{\psi_i^{(k)} \equiv (I - Q_{k-1}^a)\phi_i^{(k)} : \quad \forall x_i \in \mathcal{N}_k^{(1)}\}$$

The corresponding preconditioners can be constructed by repeating the procedure discussed in section 2 or section 3.3. A spectral analysis can be established along the way presented in section 4. Details follow in the rest of this section.

5.1. *Preconditioners*

Consider the space

$$V_k^1 \equiv (I - Q_{k-1}^a)V_k^{(1)} = (I - Q_{k-1}^a)(I_k - I_{k-1})V_k$$

and introduce the operators $A_{11}^{(k)} : V_k^1 \to V_k^1$, $A_{12}^{(k)} : V_{k-1} \to V_k^1$, and $A_{21}^{(k)} : V_k^1 \to V_{k-1}$ by using the same formulas (3.8)–(3.10). Then, the matrix $A^{(k)}$ admits a two-by-two block form (3.11), with now a different space V_k^1. Following the spirit of the Algorithm 2.1, one can construct a corresponding preconditioner $B^{(k)}$ by using approximations $B_{11}^{(k)}$ to the operators $A_{11}^{(k)}$. Assume that the spectral equivalence (3.13) holds true for those approximations with a constant $b_1 \geq 0$, independent of J and k.

5.2. *Spectral estimates*

The general result in Theorem 4.1 can be employed to yield some spectral estimates for the approximate wavelet preconditioner. The key point here is to verify its conditions (3.13), (4.3) and (4.4) in this application.

Notice that any $v^{(s)} \in V_s$ admits the following unique decomposition:

$$v^{(s)} = v^{(s)^1} + v^{(s-1)} \tag{5.2}$$

where

$$\begin{aligned} v^{(s)^1} &= (I - Q_{s-1}^a)(I_s - I_{s-1})v^{(s)} \in V_s^1 \equiv (I - Q_{s-1}^a)V_s^{(1)} \\ v^{(s-1)} &= Q_{s-1}^a v^{(s)} + (I - Q_{s-1}^a)I_{s-1}v^{(s)} \in V_{s-1} \end{aligned} \tag{5.3}$$

The above relation provides the terms $v^{(s)^1}$ and $v^{(s-1)}$ in (4.3) and (4.4) with $v^{(k)} = v \in V_k$.

Let $e_s = v^{(s)} - Q_s v$ be the deviation of $v^{(s)}$ from $Q_s v$. Since Q_s^a is an approximation of Q_s, it is reasonable to believe that the deviation e_s can be essentially neglected in the argument. The following lemma provides a rigorous estimate on this perturbation.

Lemma 5.1. *One has the following identity:*

$$e_{s-1} = [Q_{s-1} + R_{s-1}]e_s + R_{s-1}(Q_s - Q_{s-1})v \tag{5.4}$$

where $R_{s-1} = (Q_{s-1} - Q_{s-1}^a)(I_{s-1} - I_s)$.

Proof It can be seen that

$$\begin{aligned} e_{s-1} &= v^{(s-1)} - Q_{s-1}v \\ &= (Q_{s-1} - Q_{s-1}^a)I_{s-1}v^{(s)} + Q_{s-1}^a v^{(s)} - Q_{s-1}v \\ &= (Q_{s-1} - Q_{s-1}^a)I_{s-1}(v^{(s)} - Q_s v) + Q_{s-1}^a(v^{(s)} - Q_s v) \\ &\quad +(Q_{s-1} - Q_{s-1}^a)I_{s-1}Q_s v + Q_{s-1}^a Q_s v - Q_{s-1}Q_s v \\ &= (Q_{s-1} - Q_{s-1}^a)I_{s-1}e_s + Q_{s-1}^a I_s e_s \\ &\quad +Q_{s-1}(I_{s-1}Q_s v - Q_s v) - Q_{s-1}^a(I_{s-1}Q_s v - Q_s v) \\ &= (Q_{s-1} - Q_{s-1}^a)(I_{s-1} - I_s)e_s + (Q_{s-1} - Q_{s-1}^a)e_s + Q_{s-1}^a e_s \\ &\quad +(Q_{s-1} - Q_{s-1}^a)(I_{s-1} - I_s)Q_s v \\ &= \left[Q_{s-1} + (Q_{s-1} - Q_{s-1}^a)(I_{s-1} - I_s)\right] e_s \\ &\quad +(Q_{s-1} - Q_{s-1}^a)(I_{s-1} - I_s)Q_s v \end{aligned}$$

This together with the fact that $(I_{s-1} - I_s)Q_{s-1} = 0$ completes the proof of the lemma. ∎

Using now (5.1) and the L^2-boundedness of the nodal interpolation operators $I_{s-1} : V_s \to V_{s-1}$ we arrive at

$$\|R_{s-1}v\|_0 \le C_R\tau\|v\|_0 \qquad \forall v \in V_s \tag{5.5}$$

for some constant C_R. It follows from (5.4) and (5.5) that

$$\|e_{s-1}\|_0 \le (1 + C_R\tau)\|e_s\|_0 + C_R\tau\|(Q_s - Q_{s-1})v\|_0 \tag{5.6}$$

From now on we assume that τ is sufficiently small such that

$$C_R\tau \le q_1 = \text{Const} < 1 \tag{5.7}$$

It is then trivial to see that

$$(1 + C_R\tau)\frac{1}{2} \le q = \frac{1 + q_1}{2} = \text{Const} < 1 \tag{5.8}$$

Observe that $e_k = 0$. Then, a recursive use of (5.6) leads to

$$\|e_{s-1}\|_0 \le C_R\tau \sum_{j=s}^{k}(1 + C_R\tau)^{j-s}\|(Q_j - Q_{j-1})v\|_0$$

Therefore, with $h_j = \frac{1}{2}h_{j-1}$,

$$\begin{aligned}
\|e_{s-1}\|_0 &\le C_R\tau h_{s-1}\sum_{j=s}^{k}(1 + C_R\tau)^{j-s}h_s^{-1}\|(Q_j - Q_{j-1})v\|_0 \\
&= C_R\tau h_{s-1}\sum_{j=s}^{k}(1 + C_R\tau)^{j-s}h_s^{-1}h_j h_j^{-1}\|(Q_j - Q_{j-1})v\|_0 \\
&= C_R\tau h_{s-1}\sum_{j=s}^{k}(1 + C_R\tau)^{j-s}\left(\tfrac{1}{2}\right)^{j-s} h_j^{-1}\|(Q_j - Q_{j-1})v\|_0 \\
&\le C_R\tau h_{s-1}\sum_{j=s}^{k} q^{j-s}h_j^{-1}\|(Q_j - Q_{j-1})v\|_0 \\
&\le C_R\tau h_{s-1}\frac{1}{\sqrt{1-q}}\left[\sum_{j=s}^{k} q^{j-s}h_j^{-2}\|(Q_j - Q_{j-1})v\|_0^2\right]^{\frac{1}{2}}
\end{aligned}$$

The last inequality shows

$$\begin{aligned}
\sum_{s=1}^{k} h_{s-1}^{-2}\|e_{s-1}\|_0^2 &\le C_R^2\tau^2\frac{1}{1-q}\sum_{s=1}^{k}\sum_{j=s}^{k} q^{j-s}h_j^{-2}\|(Q_j - Q_{j-1})v\|_0^2 \\
&\le C_R^2\tau^2\frac{1}{(1-q)^2}\sum_{j=1}^{k} h_j^{-2}\|(Q_j - Q_{j-1})v\|_0^2
\end{aligned} \tag{5.9}$$

The above inequality will turn out to be very useful in the spectral analysis.

We point out that Lemma 4.3 is valid for the operator $A_{11}^{(k)}$ obtained from the approximate wavelet basis. Thus, it is preferable to choose a diagonal preconditioner $B_{11}^{(s)} = \alpha h_s^{-2} I$ for $A_{11}^{(s)}$, where α should be adjusted to satisfy (3.13).

Lemma 5.2. *If (a.i) is valid and τ is sufficiently small (but fixed), then (4.3) holds true for some constant ϱ_1.*

Proof The component $v^{(s)^1}$ is given by (5.3). Since

$$v^{(s)^1} = v^{(s)} - v^{(s-1)} = e_s + Q_s v - e_{s-1} - Q_{s-1} v$$

then

$$\|v^{(s)^1}\|_0 \le \|(Q_s - Q_{s-1})v\|_0 + \|e_s\|_0 + \|e_{s-1}\|_0 \tag{5.10}$$

Notice that

$$\sum_{s=1}^{k} (A_{11}^{(s)} v^{(s)^1}, v^{(s)^1}) \le C \sum_{s=1}^{k} h_s^{-2} \|v^{(s)^1}\|_0^2$$

Thus, from (5.10)

$$\begin{aligned}
\sum_{s=1}^{k} (A_{11}^{(s)} v^{(s)^1}, v^{(s)^1}) \quad & \le C \sum_{s=1}^{k} h_s^{-2} \left(\|(Q_s - Q_{s-1})v\|_0 + \|e_s\|_0 + \|e_{s-1}\|_0\right)^2 \\
& \le C \sum_{s=1}^{k} h_s^{-2} \|(Q_s - Q_{s-1})v\|_0^2 + C \sum_{s=0}^{k-1} h_s^{-2} \|e_s\|_0^2 \\
& \le C(\tau) \sum_{s=0}^{k} h_s^{-2} \|(Q_s - Q_{s-1})v\|_0^2 \le C a(v, v)
\end{aligned}$$

Here we have used the estimates (5.9) and (a.i). This completes the proof of the lemma. ■

Lemma 5.3. *If (a.i) and (a.ii) are valid and τ is sufficiently small (but fixed), then (4.4) holds true for some constant ϱ_2.*

Proof With the choice of $B_{11}^{(s)^1} = \alpha h_s^{-2} I$, one has

$$\sum_{s=1}^{k} (B_{11}^{(s)^{-1}} A_{12}^{(s)} v^{(s-1)}, A_{12}^{(s)} v^{(s-1)}) \le C \sum_{s=1}^{k} h_s^2 \|A_{12}^{(s)} v^{(s-1)}\|_0^2$$

Using the inequality prior to (4.8) and the fact that $v^{(s-1)} = e_{s-1} + Q_{s-1} v$ one obtains

$$\begin{aligned}
\sum_{s=1}^{k} (B_{11}^{(s)^{-1}} A_{12}^{(s)} v^{(s-1)}, A_{12}^{(s)} v^{(s-1)}) \quad & \le C \sum_{s=1}^{k} h_s^2 \|A^{(s)} v^{(s-1)}\|_0^2 \\
& \le C \sum_{s=1}^{k} h_s^2 \left(\|A^{(s)} e_{s-1}\|_0^2 + \|A^{(s)} Q_{s-1} v\|_0^2\right) \\
& \le C \sum_{s=1}^{k} h_{s-1}^{-2} \|e_{s-1}\|_0^2 + C \sum_{s=1}^{k} h_s^2 \|A^{(s)} Q_{s-1} v\|_0^2
\end{aligned} \tag{5.11}$$

The first sum in the last line of (5.11) can be estimated by using (5.9) and (a.i), yielding

$$\sum_{s=1}^{k} h_{s-1}^{-2} \|e_{s-1}\|_0^2 \le Ca(v, v)$$

The second sum in the last line of (5.11) has been estimated in Lemma 4.2. This completes the proof of the lemma. ■

To summarize, the following main result has been proved:

Theorem 5.1. *Let $B^{(k)}$ be the approximate wavelet preconditioner constructed by using Algorithm 2.1 or 3.1.*

1. *In addition to (a.i) and (a.ii), assume that the approximate L^2-projections Q_k^a are sufficiently close to the exact L^2-projections Q_k so that (5.1) is valid with the constraint (5.7). Then, $B^{(k)}$ is spectrally equivalent to the operator $A^{(k)}$.*
2. *Without assuming the inequality (a.ii), the preconditioner $B^{(k)}$ is nearly spectrally equivalent to the solution operator $A^{(k)}$ for sufficiently small τ. More precisely, an analogue of Theorem 4.3 or Theorem 4.4 holds true.*

6. Stability analysis

Our objective in this section is to show that the finite element discretization matrix for the second-order elliptic operator is well-conditioned with respect to the approximate wavelet basis. A spectral estimate for the additive preconditioner will be presented as well.

6.1. Some norm equivalence

The goal here is to verify the well-conditionedness of the matrix $A_{11}^{(k)}$ that was claimed in Lemma 4.3 for the wavelet and approximate wavelet bases. Let us first establish a norm equivalence for the modified hierarchical basis functions discussed in section 3.2.

Lemma 6.1. *Let $V_k^1 = (I - M_{k-1})V_k^{(1)}$ be the modified hierarchical subspace of level k. Then, there are constants c_1 and c_2 independent of k such that for any $\psi^1 = (I - M_{k-1})\phi^1 \in V_k^1$, with $\phi^1 \in V_k^{(1)}$,*

$$c_1\|\phi^1\|_i^2 \le \|\psi^1\|_i^2 \le c_2\|\phi^1\|_i^2, \qquad i = 0, 1 \tag{6.1}$$

Recall that $\|.\|_s$ stands for the norm in the Sobolev space $H^s(\Omega)$, $s = 0, 1$.

Proof The following strengthened Cauchy inequality is valuable: there exists a constant $\gamma \in (0, 1)$, independent of the mesh size or the level index k such that

$$(\nabla\phi^1, \nabla\tilde{\phi}) \le \gamma(\nabla\phi^1, \nabla\phi^1)^{\frac{1}{2}} (\nabla\tilde{\phi}, \nabla\tilde{\phi})^{\frac{1}{2}}, \qquad \forall\phi^1 \in V_k^{(1)}, \tilde{\phi} \in V_{k-1} \tag{6.2}$$

In fact, we shall make use of the following version of (6.2):

$$(\nabla(\phi^1 + \tilde{\phi}), \nabla(\phi^1 + \tilde{\phi})) \ge (1 - \gamma^2)(\nabla\phi^1, \nabla\phi^1), \qquad \forall\phi^1 \in V_k^{(1)}, \tilde{\phi} \in V_{k-1} \tag{6.3}$$

A derivation of (6.2) and (6.3) can be found from Bank and Dupont [3], Braess [6] or Axelsson and Gustafsson [1].

We first establish (6.1) for the case $i = 1$. With $\tilde{\phi} = -M_{k-1}\phi^1$ we see from (6.3) that

$$(1-\gamma^2)\|\phi^1\|_1^2 \leq \|\psi^1\|_1^2$$

Thus, the inequality on the left-hand side of (6.1) is valid with $c_1 = 1-\gamma^2$. To derive the part on the right-hand side, we use the standard inverse inequality to obtain

$$\|\psi^1\|_1^2 \leq Ch_k^{-2}\|\psi^1\|_0^2 \leq Ch_k^{-2}\|\phi^1\|_0^2$$

where we have used the L^2-boundedness of the linear operator M_{k-1}. Observe now that since $\phi^1 \in V_k^{(1)}$, there exists a constant C such that

$$\|\phi^1\|_0^2 \leq Ch_k^2\|\phi^1\|_1^2 \tag{6.4}$$

It follows that $\|\psi^1\|_1^2 \leq C\|\phi^1\|_1^2$ for some constant C. This completes the proof of (6.1) for $i = 1$. Similar arguments can be applied to verify the case $i = 0$. ■

Proof (Proof of Lemma 4.3) For any $\psi^1 = (I - M_{k-1})\phi^1 \in V_k^1$, since

$$(A_{11}^{(k)}\psi^1, \psi^1) = a(\psi^1, \psi^1)$$

and the bilinear form $a(\cdot,\cdot)$ is equivalent to the H^1-inner product, then there are positive constants τ_i such that

$$\tau_1\|\psi^1\|_1^2 \leq (A_{11}^{(k)}\psi^1, \psi^1) \leq \tau_2\|\psi^1\|_1^2$$

Using the norm equivalence (6.1), (6.4) and the inverse inequality we obtain with other positive constants $\tilde{\tau}_i$,

$$\tilde{\tau}_1 h_k^{-2}\|\phi^1\|_0^2 \leq (A_{11}^{(k)}\psi^1, \psi^1) \leq \tilde{\tau}_2 h_k^{-2}\|\phi^1\|_0^2$$

The above inequalities verify the validity of Lemma 4.3. ■

6.2. The H^1-stability of the approximate wavelet basis

For any $v \in V$ let

$$v = \sum_{x_i \in \mathcal{N}_0} c_{0,i}\phi_i^{(0)} + \sum_{k=1}^{J} \sum_{x_i \in \mathcal{N}_k^{(1)}} c_{k,i}(I - Q_{k-1}^a)\phi_i^{(k)} \tag{6.5}$$

be its representation with respect to the approximate wavelet basis. The corresponding coefficient norm of v is given by (1.3). Our main result in this section is the following norm equivalence:

Theorem 6.1. *There exists a small (but fixed) $\tau_0 > 0$ such that if the approximate wavelet basis satisfies (5.1) with $\tau \in (0, \tau_0)$, then there are constants c_1 and c_2 satisfying*

$$c_1|||v|||^2 \leq \|v\|_1^2 \leq c_2|||v|||^2 \qquad \forall v \in V \tag{6.6}$$

From now on, the above equivalence relation will be abbreviated as $|||v|||^2 \simeq \|v\|_1^2$.

Proof We first rewrite (6.5) as follows:

$$v = \sum_{k=0}^{J} v^{(k)^1} \tag{6.7}$$

where, with $Q_{-1}^a = 0$,

$$v^{(k)^1} = \sum_{x_i \in \mathcal{N}_k^{(1)}} c_{k,i}(I - Q_{k-1}^a)\phi_i^{(k)} \in V_k^1 \tag{6.8}$$

Furthermore, by letting $\phi^{(k)} = \sum_{x_i \in \mathcal{N}_k^{(1)}} c_{k,i}\phi_i^{(k)} \in V_k^{(1)}$ we see that $v^{(k)^1} = (I - Q_{k-1}^a)\phi^{(k)}$. Thus, by using (6.1) in Lemma 6.1 (with $i = 0$ and $M_{k-1} = Q_{k-1}^a$) we obtain

$$\|\phi^{(k)}\|_0^2 \simeq \|v^{(k)^1}\|_0^2 \tag{6.9}$$

Since $\phi^{(k)} \in V_k^{(1)}$, then

$$\|\phi^{(k)}\|_0^2 \simeq h_k^d \sum_{x_i \in \mathcal{N}_k^{(1)}} c_{k,i}^2$$

Combining the above with (6.9) yields

$$|||v|||^2 \simeq \sum_{k=0}^{J} h_k^{-2}\|v^{(k)^1}\|_0^2$$

This, together with Lemma 6.2 below, completes the proof of the lemma. ■

Lemma 6.2. *Let v and $v^{(k)^1}$ be related as in (6.7), (6.8). If the condition of Theorem 6.1 is satisfied, then*

$$\|v\|_1^2 \simeq \sum_{k=0}^{J} h_k^{-2}\|v^{(k)^1}\|_0^2 \tag{6.10}$$

Proof The proof of Lemma 5.2 also shows that

$$\sum_{k=0}^{J} h_k^{-2}\|v^{(k)^1}\|_0^2 \le C\|v\|_1^2$$

Thus, it suffices to establish the following inequality:

$$\|v\|_1^2 \le C\sum_{k=0}^{J} h_k^{-2}\|v^{(k)^1}\|_0^2 \tag{6.11}$$

For the inner product

$$b(v, w) \equiv (\nabla v, \nabla w)$$

the following analogue of (a.ii) is valid,

$$\left|b(\psi_i, \psi_j)\right|^2 \leq \sigma \delta^{2(j-i)} h_j^{-2} b(\psi_i, \psi_i) \|\psi_j\|_0^2 \qquad \forall \psi_i \in V_i,\ \psi_j \in V_j,\ j \geq i$$

which, based on the inverse inequality gives,

$$|b(\psi_i, \psi_j)| \leq C\sqrt{\sigma}\delta^{(j-i)} h_j^{-1} h_i^{-1} \|\psi_i\|_0 \, \|\psi_j\|_0 \qquad \forall \psi_i \in V_i,\ \psi_j \in V_j,\ j \geq i$$

Substituting the above into $\|v\|_1^2 = b(v, v) = \sum_{j,k=0}^{J} b(v^{(j)^1}, v^{(k)^1})$ yields,

$$\begin{aligned}
\|v\|_1^2 &\leq C \sum_{j,k=0}^{J} \sqrt{\sigma}\delta^{|j-k|} h_j^{-1} h_k^{-1} \|v^{(j)^1}\|_0 \, \|v^{(k)^1}\|_0 \\
&\leq C\sqrt{\sigma} \left(\sum_{j,k=0}^{J} \delta^{|j-k|} h_j^{-2} \|v^{(j)^1}\|_0^2 \right)^{\frac{1}{2}} \left(\sum_{j,k=0}^{J} \delta^{|j-k|} h_k^{-2} \|v^{(k)^1}\|_0^2 \right)^{\frac{1}{2}} \\
&= C\sqrt{\sigma} \sum_{j,k=0}^{J} \delta^{|j-k|} h_j^{-2} \|v^{(j)^1}\|_0^2 \\
&\leq C\sqrt{\sigma} \frac{1+\delta}{1-\delta} \sum_{j=0}^{J} h_j^{-2} \|v^{(j)^1}\|_0^2
\end{aligned}$$

The last inequality verifies (6.11) and, therefore, completes the proof of the lemma. ■

Remark 1. One can alternatively use the following characterization of the H_0^1-norm of the finite element space V due to Oswald [19]:

$$\|v\|_1^2 \simeq \inf_{v=\sum_{k=0}^{J} v_k,\ v_k \in V_k} \sum_{k=0}^{J} h_k^{-2} \|v_k\|_0^2$$

Therefore, for the particular decomposition $v = v^{(0)} + \sum_{k=1}^{J} v^{(k)^1}$, one immediately gets the desired upper estimate (6.11).

Since the two bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are equivalent, then we have from (6.6) that

$$c_1 |\!|\!| v |\!|\!|^2 \leq a(v, v) \leq c_2 |\!|\!| v |\!|\!|^2 \tag{6.12}$$

for some constants c_1 and c_2. The equivalent relation (6.12) can be re-interpreted as follows:

Theorem 6.2. *If the conditions of Theorem 6.1 hold true, then the matrix representation of $a(\cdot, \cdot)$ by using the approximate wavelet basis is well-conditioned.*

6.3. *On the additive preconditioner*

The additive preconditioner corresponding to the approximate wavelet basis was defined in section 2 (see Algorithm 2.2). It can also be interpreted by the following quadratic form:

$$(B_a v, v) \equiv \sum_{s=1}^{J} (B_{11}^{(s)} v^{(s)^1}, v^{(s)^1}) + (A^{(0)} v^{(0)}, v^{(0)})$$

where $v^{(s)^1}$ is the component of $v \in V$ in the subspace V_s^1 (see equations (6.5) and (6.5a), (6.5b) for more detail).

Theorem 6.3. *If the conditions of Theorem 6.1 hold true, then the additive preconditioner B_a is spectrally equivalent to the global stiffness matrix A for the bilinear form $a(\cdot, \cdot)$.*

Proof Recall that each preconditioner $B_{11}^{(s)}$ is a matrix that is spectrally equivalent to the diagonal matrix $h_s^{-2} I$. Thus,

$$(B_a v, v) \simeq \sum_{s=0}^{J} h_s^{-2} \| v^{(s)^1} \|_0^2$$

The above equivalence along with (6.10) asserts the conclusion of the theorem. ■

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